Exceptional points and dissipative phase transition

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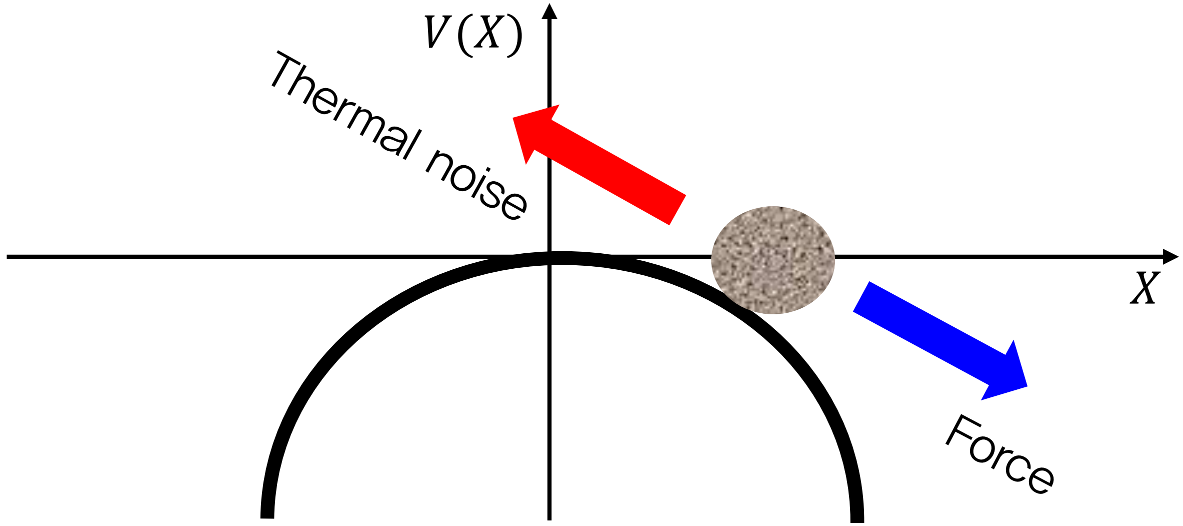
Introduction: Dissipative phase transition

- Can the thermal noise stabilize the (quantum) rock?



Fluctuation – Dissipation theorem: Thermal noise induces dissipation to the system

Phase 1	$\eta > F$ / $\eta < F$	Phase 2
Rock falls forever (Unstable)		Rock stops falling (Stable)

η: Thermal noise strength

$\eta = F$: Transition point

F: Force

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Introduction

- Lindblad equation: <u>Approximated</u> dynamical equation of the dissipative quantum system [1]

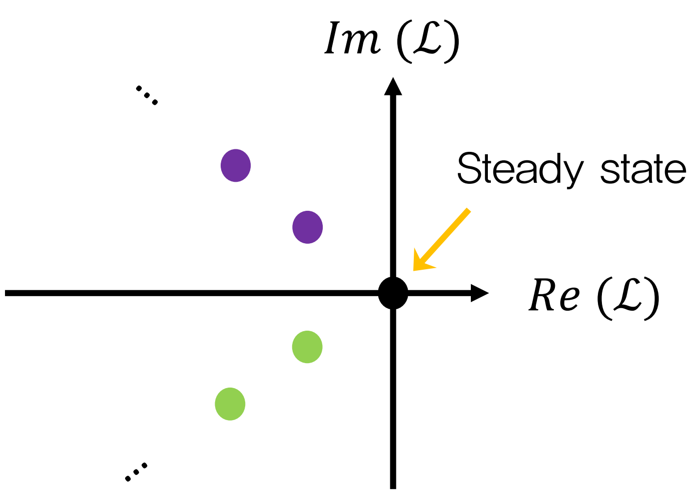
$$\frac{d\rho}{dt} = \mathcal{L}\rho = -i[H,\rho] + \sum_i \Gamma_i \left(2L_i\rho L_i^{+} - L_i^{+}L_i\rho - \rho L_i^{+}L_i\right)$$

L_i: Jump operators, Γ_i: Dissipative strengths $\propto \eta^2$

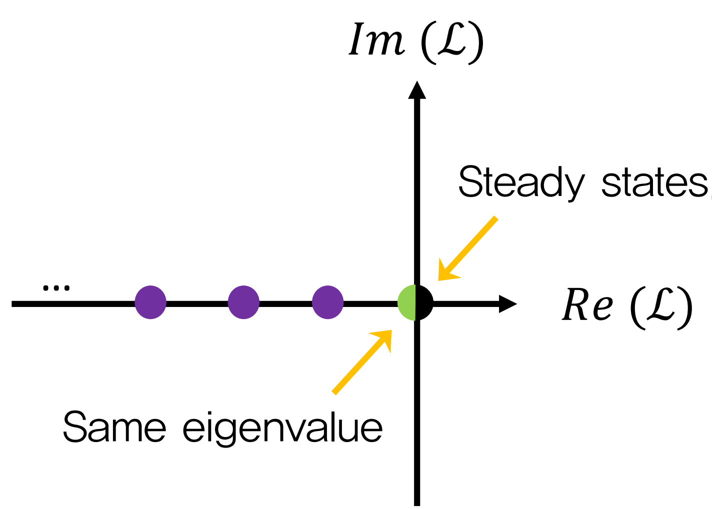
Approximation holds only when $\eta \ll F$!

- Eigenspectrum (value) of Lindbladian

$\eta > F$: stable



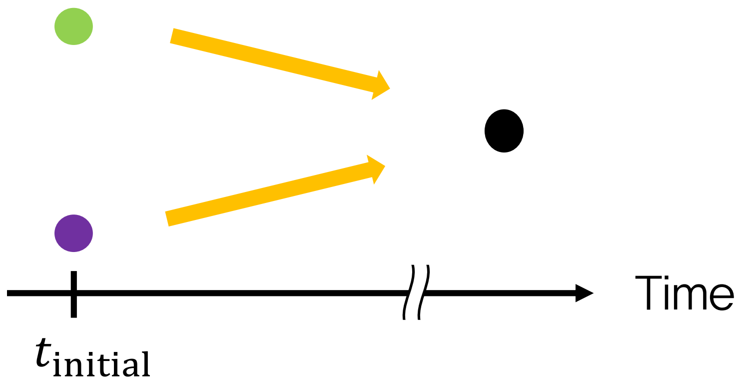
$\eta = F$ (critical point)



$\eta < F$:
Unstable phase

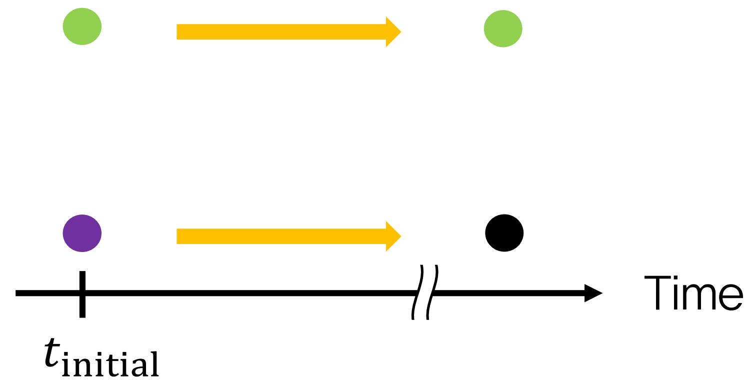
Main Question

- Unique steady state



Initial state <u>**independent**</u> steady state

- Multiple steady states

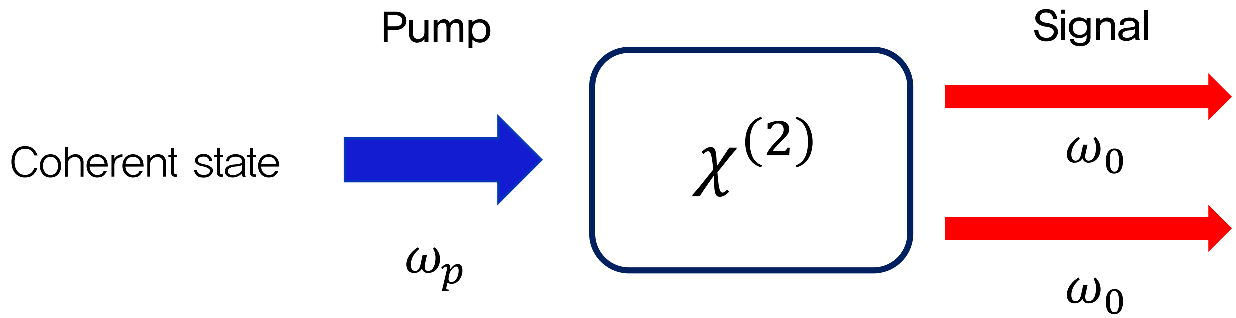


Initial state <u>**dependent**</u> steady states

- Can we expect multiple steady states in the exact dissipative equation?
- Do multiple steady states always imply the initial state dependence?
- Is it a fair evaluation to characterize the dissipative phase transition only using the eigenvalues of Lindbladian? What about its eigenstates?

Model

- Quantum squeezing Hamiltonian from 2nd order nonlinear optical interaction [1]



a, a^+: Bosons
$[a, a^+] = 1$

$$H = \omega_0\left(a^+a + \frac{1}{2}\right) + \frac{\lambda}{2}(a^{+2}e^{i\phi} + a^2e^{-i\phi})$$

Site frequency

Interaction strength of photons

Phase difference of photons

Normal mode transformation

$$H = \frac{1}{2}(\omega_0 + \lambda)P^2 + \frac{1}{2}(\omega_0 - \lambda)X^2$$

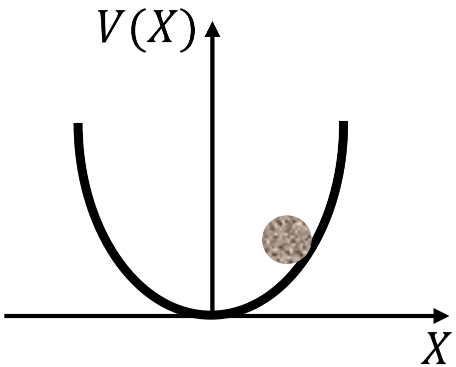
Quadratures (not real position and momentum!)

$$X := \frac{i}{\sqrt{2}}(e^{-i\phi/2}a - e^{i\phi/2}a^+) \qquad P := \frac{i}{\sqrt{2}}(e^{-i\phi/2}a + e^{i\phi/2}a^+) \qquad [X, P] = i$$

Model

- Quantum harmonic oscillators

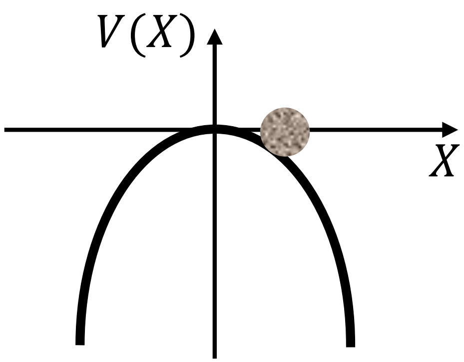
$\omega_0 > \lambda$: Simple Harmonic Oscillator (SHO)



$$\frac{d^2X}{dt^2} + \omega^2 X = 0$$

$$X(t) = X(0)\cos\omega t + P(0)\sin\omega t$$

$\omega_0 < \lambda$: Inverted Harmonic Oscillator (IHO)



$$\frac{d^2X}{dt^2} - \omega^2 X = 0$$

$$X(t) = X(0)\cosh\omega t + P(0)\sinh\omega t$$

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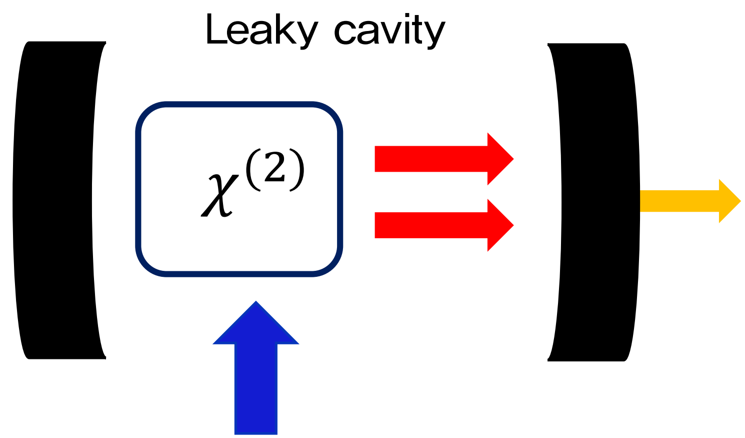
Model in the dissipative cavity

$$H_{\text{sys}} = \omega_0\left(a^+a + \frac{1}{2}\right) + \frac{\lambda}{2}\left(a^{+2}e^{i\phi} + a^2e^{-i\phi}\right)$$



$$H_{\text{sys-bath}} = \sum_k g_k b_k a^+ + H.C.$$ → Rotating wave approximated interaction

$$H_{\text{bath}} = \sum_k \Omega_k\left(b_k^+ b_k + \frac{1}{2}\right)$$ → Harmonic oscillator bath (cavity)

$$[b_k, b_l^+] = \delta_{kl}$$

- Z_2 Symmetry of the model

Z_2 operation on the operator X → $Z_2[X] \coloneqq e^{i\pi a^+a} X e^{-i\pi a^+a}$

$$Z_2[H + H_{\text{bath}}] = H + H_{\text{bath}} \qquad Z_2[H_{\text{sys-bath}}] = -H_{\text{sys-bath}}$$
$$Z_2[H + H_{\text{bath}} + H_{\text{sys-bath}}] \neq H + H_{\text{bath}} + H_{\text{sys-bath}}$$

→ Interaction Hamiltonian breaks Z_2 symmetry

Dissipative dynamics

- Compute the coherent state path integral to obtain exact Liouville equation [2]

$$\rho(\alpha_f,\alpha_f';t) = \langle \alpha_f|\rho(t)|\alpha_f'\rangle = \int d\chi(\alpha_i)d\chi(\alpha_i')\, K(\alpha_f,\alpha_f';t|\alpha_i,\alpha_i';0)\,\rho(\alpha_i,\alpha_i';0)$$

Propagating functional — Initial state

$$\Rightarrow \frac{d\rho}{dt} = \mathcal{L}\rho = -i[H,\rho] + 2\Gamma\left(a\rho a^+ - \frac{1}{2}\{a^+a,\rho\}\right) + 2\Gamma N(a^+\rho a + a\rho a^+ - a^+a\rho - \rho a a^+)$$

Equation holds for arbitrary $\Gamma \le \sqrt{\lambda^2-\omega^2}$

Liouvillian is Z_2 symmetric, $Z_2[\mathcal{L}[\rho]] = \mathcal{L}[Z_2[\rho]]$

- Langevin equation: stationary path equation (equation for least action) + fluctuations

$$\frac{d}{dt}\begin{pmatrix} a(t) \\ a^+(t)\end{pmatrix} = \begin{pmatrix} -i\tilde{\omega}-\Gamma & -i\lambda e^{i\phi} \\ i\lambda e^{-i\phi} & i\tilde{\omega}-\Gamma\end{pmatrix}\begin{pmatrix} a(t) \\ a^+(t)\end{pmatrix} - \sum_k \begin{pmatrix} \eta_k(t) \\ \eta_k^+(t)\end{pmatrix}$$

($\tilde{\omega}$: Renormalized frequency $(\omega_0-\Delta)$)

$$\frac{\delta K}{\delta\alpha} = \frac{\delta K}{\delta\alpha^*} = 0$$

Gaussian white noise

Dissipative dynamics

- Z_2 Symmetry of order parameter, $\langle a \rangle_{ss} = \lim_{t\to\infty} \mathrm{Tr}[a(t)\rho]$ [3]

$$\langle a \rangle_{ss} = \mathrm{Tr}[a_{ss}\rho] = \mathrm{Tr}[a\rho_{ss}] = -\mathrm{Tr}\left[aZ_2[\rho_{ss}]\right] \Rightarrow \begin{cases} = 0, Z_2[\rho_{ss}] = \rho_{ss} \\ \neq 0, Z_2[\rho_{ss}] \neq \rho_{ss} \end{cases}$$

- Evolution matrix of the first moments

$$\frac{d}{dt}\begin{pmatrix} \langle a \rangle \\ \langle a^{+} \rangle \end{pmatrix} = \begin{pmatrix} -i\tilde{\omega} - \Gamma & -i\lambda e^{i\phi} \\ i\lambda e^{-i\phi} & i\tilde{\omega} - \Gamma \end{pmatrix}\begin{pmatrix} \langle a \rangle \\ \langle a^{+} \rangle \end{pmatrix}$$

→ Parity–Time reversal (PT) symmetric time evolution matrix [4] → Eigenvalues $E_{\pm} = -\Gamma \pm \sqrt{\lambda^2 - \tilde{\omega}^2}$

Simple Harmonic

1) $\tilde{\omega} > \lambda$ (PT unbroken)

→ Stable ($\mathrm{Re}\ E_{+} < 0$)

$$\begin{pmatrix} \langle a \rangle(t) \\ \langle a^{+} \rangle(t) \end{pmatrix} \xrightarrow{t\to\infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Z_2 Symmetry Unbroken

Inverted Harmonic

2) $\tilde{\omega} < \lambda$ (PT broken), $\Gamma > \sqrt{\lambda^2 - \tilde{\omega}^2}$

→ Stable ($\mathrm{Re}\ E_{+} < 0$)

$$\begin{pmatrix} \langle a \rangle(t) \\ \langle a^{+} \rangle(t) \end{pmatrix} \xrightarrow{t\to\infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Z_2 Symmetry Unbroken

3) $\tilde{\omega} < \lambda$ (PT broken), $\Gamma < \sqrt{\lambda^2 - \tilde{\omega}^2}$

→ Unstable ($\mathrm{Re}\ E_{+} > 0$)

$$\begin{pmatrix} \langle a \rangle(t) \\ \langle a^{+} \rangle(t) \end{pmatrix} \xrightarrow{t\to\infty} \pm\infty$$

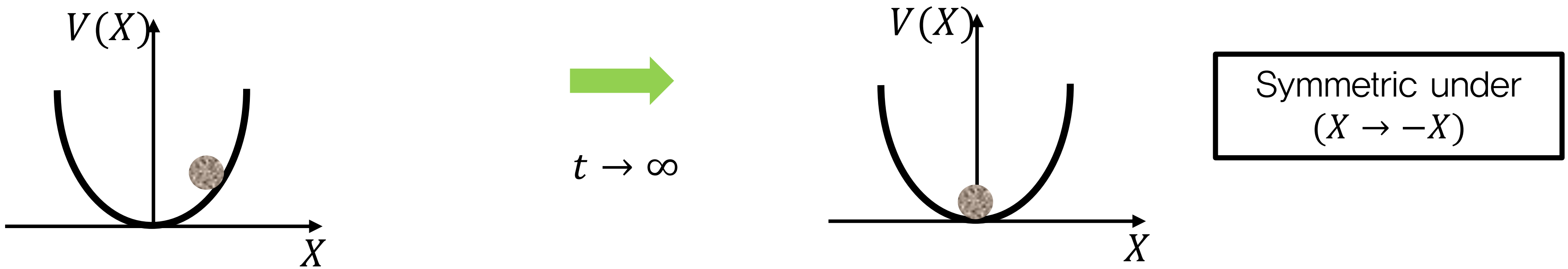
Z_2 Symmetry Broken

(Can be zero when $\begin{pmatrix} \langle a \rangle(0) \\ \langle a^{+} \rangle(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$)

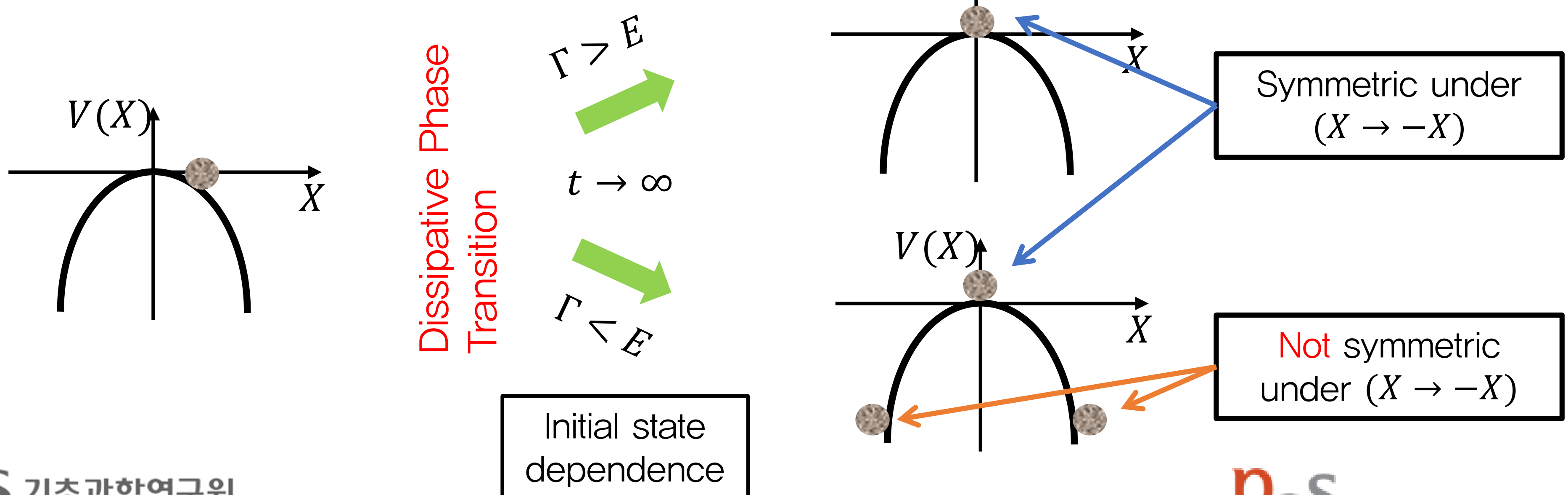
Role of Z_2 Symmetry

- Z_2 Symmetry of the system $(X \to -X)$ in the presence of dissipation

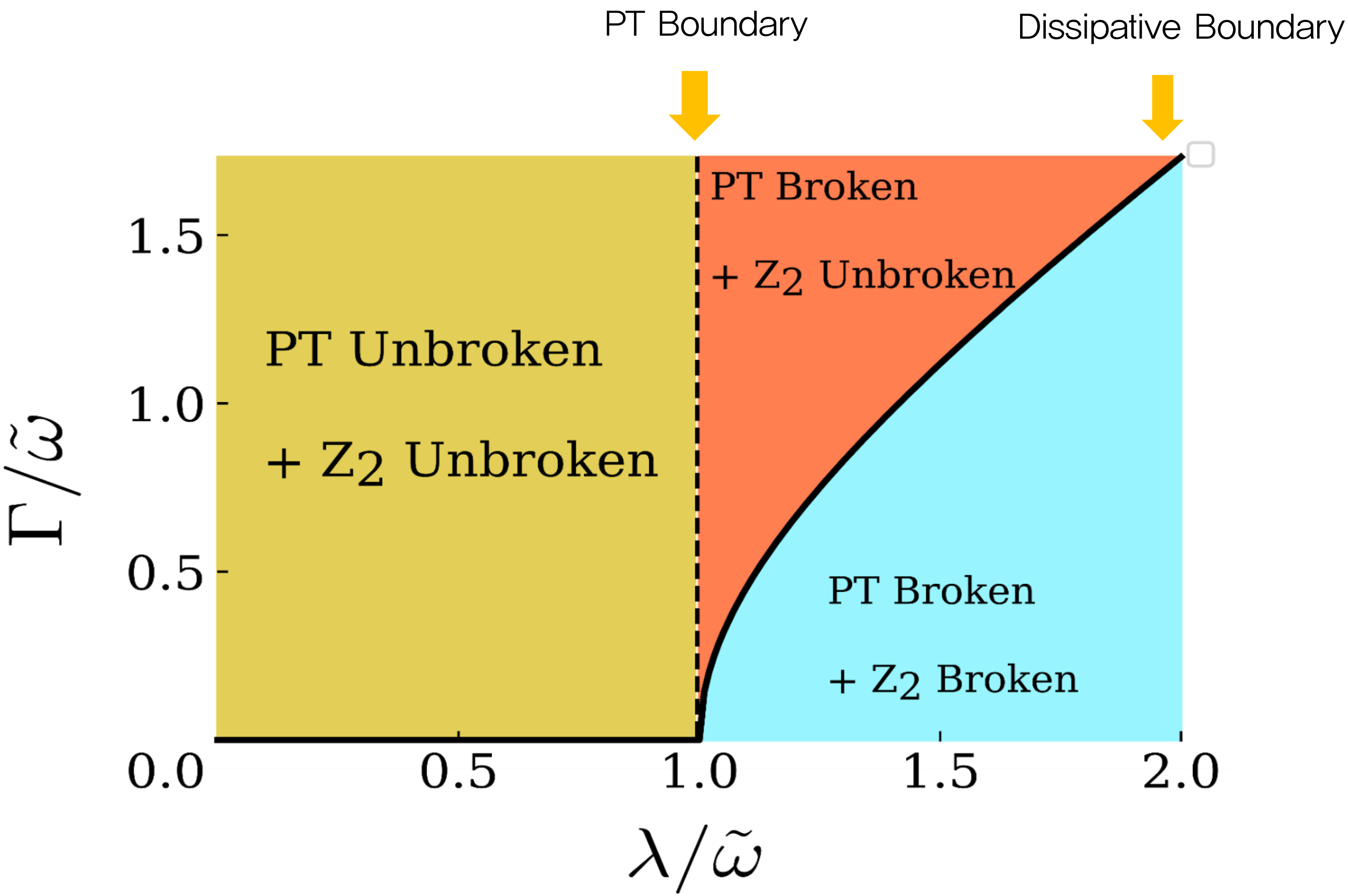
1) $\tilde{\omega} > \lambda$ (PT unbroken)



2) $\tilde{\omega} < \lambda$ (PT broken)



Dissipative Phase Diagram [5]



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Dissipative dynamics from eigenstates

- Construct steady–state from the given Gaussian initial state [6]

Covariance matrix (2nd moments)

$$C_s = \begin{pmatrix} \langle X^2(t)\rangle & \frac{1}{2}\langle X(t)P(t)+P(t)X(t)\rangle \\ \frac{1}{2}\langle X(t)P(t)+P(t)X(t)\rangle & \langle P^2(t)\rangle \end{pmatrix}$$

Symplectic matrix $S \in SP(2)$, $SC_sS^T = dI_2$

$$\rho_{ss} = \frac{\exp\left[-\beta\left(P'^2 + X'^2\right)\right]}{Z}$$

$$Z = \mathrm{Tr}\left[-\beta\left(P'^2 + X'^2\right)\right], \quad \begin{pmatrix} X' \\ P' \end{pmatrix} = S\begin{pmatrix} X \\ P \end{pmatrix},$$

$$\beta = \coth^{-1} 2d\,, \qquad [X', P'] = i$$

- Is our conclusion based on eigenvalues true?

Ex) $\lambda \neq 0,\ \tilde{\omega} = 0,\ \phi = \pi/2,\ \Gamma = \lambda$ (Phase boundary → Z_2 symmetry broken regime)

$$\rho(t \gg 1) \to \frac{\exp\left[-\beta(t)\left(P^2 + \left(\frac{X - \langle X\rangle_0}{\sigma(t)}\right)^2\right)\right]}{Z} \xrightarrow[t\to\infty]{}$$

$$\rho_{ss} = \frac{\exp[-\beta P^2]}{Z}$$

Z_2 Symmetry still Unbroken!

$$\beta(t \gg 1) \xrightarrow[t\to\infty]{} \frac{2}{2N+1} \equiv \beta, \quad \sigma^2(t \gg 1) \approx \frac{4}{2N+1}\left(\langle X^2\rangle_0 + (2N+1)\Gamma t\right) \xrightarrow[t\to\infty]{} \infty$$

Dissipative phase boundary

- What happens in the dissipative boundary?

$$\tilde{\omega} < \lambda \text{ (PT Broken)}, \ \Gamma = \sqrt{\lambda^2 - \tilde{\omega}^2}$$

Multiple steady state ?

$$\begin{pmatrix} \langle a \rangle(t) \\ \langle a^+ \rangle(t) \end{pmatrix} \xrightarrow{t \to \infty} \begin{cases} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

$\Rightarrow$

Initial state independent !

$$\begin{pmatrix} \langle a \rangle(t) \\ \langle a^+ \rangle(t) \end{pmatrix} \xrightarrow{t \to \infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{when } (\tilde{\omega} = 0)$$

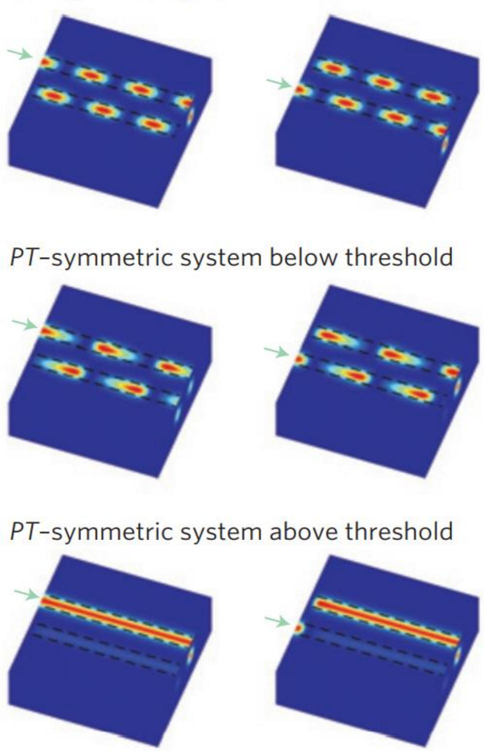
Since $\left(\langle a \rangle(t)\right)^* \neq \langle a^+ \rangle(t)$

- Multiple zero eigenvalues + states coalesce ➔ Exceptional point

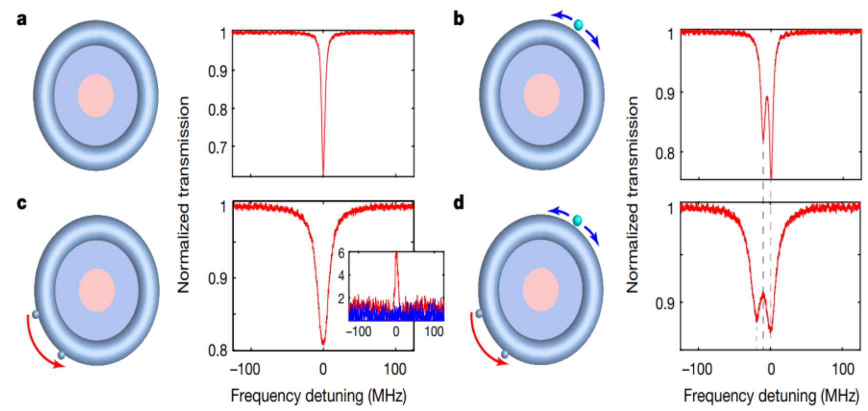
- No symmetry breaking ➔ No phase transition

Exceptional point

- Application of exceptional point



Uni–directional propagation of light [8]



Quantum sensing of nano particles [9]

What is the application of exceptional points induced by dissipation?

Conclusion

- Dissipative phase transition does not exist in the exact formulation.
- Phase boundary exhibits exceptional points (not degenerate states).

Reference

[1] The theory of open quantum systems (2002)

[2] Phys. Rev. A **52**, 815 (1995)

[3] Ann. Phys. 327(5), 1408 (2012)

[4] Phys. Rev. A **98**, 042118 (2018)

[5] Phys. Rev. A **99**, 063834 (2019)

[6] Phys. Rev. Lett. **125**, 240405 (2020)

[7] Phys. Rev. E **85**, 011126 (2012)

[8] Nat. Phys. **6**, 192–195 (2010)

[9] Nature, **548**, 192–196 (2017)

Additional information

- Block-diagonalized Liouvillian

$$Z_2[\rho_+] = \rho_+$$

$$\hat{L} = \begin{pmatrix} Z_{2+} & 0 \\ 0 & Z_{2-} \end{pmatrix} \qquad \boxed{Z_2[\rho_\pm] = \pm\rho_\pm}$$

$$Z_2[\rho_-] = -\rho_-$$

- Steady states, $\hat{L}[\rho] = \lambda\rho,\ \mathrm{Re}(\lambda) = 0$

– Even sector $\lambda \in Z_{2+}$: Physical steady state

– Odd sector $\lambda \in Z_{2-}$: Unphysical steady state (traceless)

$$\because\ \mathrm{Tr}\big[Z_2[\rho]\big] = \mathrm{Tr}\big[e^{i\pi a^+ a}\rho e^{-i\pi a^+ a}\big] = \mathrm{Tr}[\rho] = -\mathrm{Tr}[\rho]$$

- Dissipative kernel & Noise correlation (Gaussian)

$$\Sigma(t-s) \coloneqq \sum_k |g_k|^2 e^{-i\Omega_k(t-s)} = (\Gamma - i\Delta)\delta(t-s) \qquad \langle b_k^+(t) b_l(s)\rangle = (\Gamma N + i\Delta)\delta(t-s)$$